

THE BISHOP-PHELPS-BOLLOBÁS THEOREM ON BOUNDED CLOSED CONVEX SETS

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ABSTRACT. This paper deals with the *Bishop-Phelps-Bollobás property* (*BPBp* for short) on bounded closed convex subsets of a Banach space X , not just on its closed unit ball B_X . We firstly prove that the *BPBp* holds for bounded linear functionals on arbitrary bounded closed convex subsets of a real Banach space. We show that for all finite dimensional Banach spaces X and Y the pair (X, Y) has the *BPBp* on every bounded closed convex subset D of X , and also that for a Banach space Y with property (β) the pair (X, Y) has the *BPBp* on every bounded closed absolutely convex subset D of an arbitrary Banach space X . For a bounded closed absorbing convex subset D of X with positive modulus convexity we get that the pair (X, Y) has the *BPBp* on D for every Banach space Y . We further obtain that for an Asplund space X and for a locally compact Hausdorff L , the pair $(X, C_0(L))$ has the *BPBp* on every bounded closed absolutely convex subset D of X . Finally we study the stability of the *BPBp* on a bounded closed convex set for the ℓ_1 -sum or ℓ_∞ -sum of a family of Banach spaces.

1. INTRODUCTION

A remarkable result so called the *Bishop-Phelps theorem* [8] came out in 1961, which states that for every Banach space X , every linear functional on X can be approximated by norm attaining ones. In fact, they showed a more general results: Let D be a closed bounded convex subset of a real Banach space X . Then the set of support functionals of D is a norm dense subset of its dual space X^* . In other words, the set of all elements of X^* that attain their suprema on D is a norm dense subset of X^* . However, Lomonosov [20] showed in 2000 that this statement cannot be extended to general complex spaces by constructing a closed bounded convex set with no support points. From now on, we assume that X and Y are real Banach spaces without any other comment.

After a while, J. Lindenstrauss [19] studied in 1963 the denseness of norm attaining linear operators between Banach spaces, which has been a classical research topic in functional analysis since then. In particular, Bourgain [10] obtained in 1976 such a surprising results that a Banach space X has the *Bishop-Phelps property* if and only if it has the Radon-Nikodym property (*RNP* for short). We recall that a Banach space X is said to have *Bishop-Phelps property* if for every bounded closed and absolutely convex subset D of X and for every Banach space Y , the subset of $\mathcal{L}(X, Y)$ attaining their suprema in norm on D is dense in the space $\mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ is the Banach space of bounded linear operators from X into Y .

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In 1977 Stegall [23] obtained a nonlinear form of Bourgain's result: Let X be a Banach space with RNP , D be a bounded closed convex subset of X and $f : D \rightarrow \mathbb{R}$ be an upper semicontinuous bounded above function. Then for $\epsilon > 0$, there exists $x^* \in X^*$ such that $\|x^*\| < \epsilon$ and $f + x^*$, $f + |x^*|$ strongly expose D . Applying this result to a vector-valued case, he showed the following. Let X be a Banach space with RNP , D be a bounded closed convex subset of X , and Y be a Banach space. Suppose that $\varphi : D \rightarrow Y$ is a uniformly bounded function such that the function $x \rightarrow \|\varphi(x)\|$ is upper semicontinuous. Then, for $\delta > 0$, there exist $T : X \rightarrow Y$ a bounded linear operator of rank one, $\|T\| < \delta$ such that $\varphi + T$ attains its supremum in norm on D and does so at most two points.

We refer to [1] surveying most of recent results on the denseness of norm attaining linear or nonlinear mappings such as multilinear mappings, polynomials or holomorphic mappings.

On the other hand Bollobás [9] sharpened in 1970 the Bishop-Phelps theorem by dealing simultaneously with norm attaining linear functionals and their norming points, which is stated as follows. We denote by B_X and S_X the closed unit ball and sphere of X , respectively.

Theorem 1.1. [9] *Let X be a Banach space and $0 < \epsilon < 1$. Given $x \in S_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\epsilon^2}{2}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that*

$$y^*(y) = 1, \quad \|x - y\| < \epsilon, \quad \text{and} \quad \|y^* - x^*\| < \epsilon + \epsilon^2.$$

He also showed that this theorem is best possible in the following sense. For any $0 < \epsilon < 1$ there exist a Banach space X , point $x \in S_X$ and functional $f \in S_{X^*}$ such that $f(x) = 1 - (\epsilon^2/2)$, but if $y \in S_X$, $g \in S_{X^*}$ and $g(y) = 1$, then either $\|f - g\| \geq \epsilon$ or $\|x - y\| \geq \epsilon$.

Since this theorem of Bollobás is stated explicitly, we have referred it more often than the theorem of Brønsted and Rockafellar [11], a more general and earlier result than Bollobás. Using the concept of the subdifferential of a convex function it is written as follows: Suppose that f is a convex proper lower semicontinuous function on a Banach space X . Then given any point $x_0 \in \text{dom}(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in \text{dom}(f)$ and $x^* \in X^*$ such that

$$x^* \in \partial(f), \quad \|x - x_0\| \leq \frac{\epsilon}{\lambda}, \quad \text{and}, \quad \|x^* - x_0^*\| \leq \lambda.$$

In particular, the domain of ∂f is dense in $\text{dom}(f)$.

Acosta et al. [2] introduced in 2008 the following definition to study this property for linear operators between Banach spaces.

Definition 1.2. [2] A pair of Banach spaces (X, Y) is said to have the *Bishop-Phelps-Bollobás property* (*BPBp* for short) if for every $\epsilon > 0$ there are $0 < \eta(\epsilon) < 1$ and $\beta(\epsilon) > 0$ with $\lim_{\epsilon \rightarrow 0} \beta(\epsilon) = 0$ such that for all $T \in \mathcal{L}(X, Y)$ and $x_0 \in S_X$ satisfying $\|T(x_0)\| > 1 - \eta(\epsilon)$, there exist a point $u \in S_X$ and an operator $S \in \mathcal{L}(X, Y)$ that satisfy the following conditions:

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \beta(\epsilon), \quad \text{and} \quad \|T - S\| < \epsilon.$$

Since they characterized in [2] the Banach space Y for which the *BPBp* holds for operators from ℓ_1 into Y , lots of interest has been caused in this property (for instance see [3, 4, 5, 6, 7, 14, 17, 18]).

We note that the *BPBp* is not so closely related with *RNP* as the *Bishop-Phelps-Bollobás property*. For example, ℓ_1 has *RNP*, but there exists a Banach space Y such that the pair (ℓ_1, Y)

does not have the *BPBp* ([2]). On the other hand, the pair $(L_1[0, 1], L_\infty[0, 1])$ has the *BPBp* ([7]), but $L_1[0, 1]$ does not have *RNP*.

So far, the *BPBp* has been studied on the closed unit ball B_X , but in this paper we deal with this property on bounded closed convex subsets D of a Banach space X , not just on B_X . We introduce the following more general definition. Let

$$\|T\|_D = \sup\{\|Tx\| : x \in D\}$$

for $T \in \mathcal{L}(X, Y)$.

Definition 1.3. Let X and Y be Banach spaces. Let D be a bounded closed convex subset of X . We say that (X, Y) has the *Bishop-Phelps-Bollobás property* on D (*BPBp* on D for short) if for every $\epsilon > 0$, there is $\eta_D(\epsilon) > 0$ such that for every $T \in L(X, Y)$, $\|T\|_D = 1$ and every $x \in D$ satisfying

$$\|T(x)\| > 1 - \eta_D(\epsilon),$$

there exist $S \in L(X, Y)$ and $z \in D$ such that

$$\|S(z)\| = 1 = \|S\|_D, \quad \|x - z\| < \epsilon \quad \text{and} \quad \|T - S\| < \epsilon.$$

Similarly we say that (X, Y) has the *Bishop-Phelps property* on D (*BPP* on D for short) if for every $\epsilon > 0$ and for every $T \in \mathcal{L}(X, Y)$ with $\|T\|_D = 1$, then there exist $S \in \mathcal{L}(X, Y)$ and $z \in D$ such that

$$\|S(z)\| = 1 = \|S\|_D \quad \text{and} \quad \|T - S\| < \epsilon.$$

In general, we cannot expect the same results in the *BPBp* on a closed bounded convex set D as those on B_X . For a uniformly convex space X the pair (X, Y) has the *BPBp* on B_X for every Banach space Y ([6, 18]). However, there is a Banach space Y such that (ℓ_2^2, Y) fails to have the *BPBp* on $D = B_{\ell_1^2}$, even though ℓ_2^2 is a uniformly convex space of dimension 2. We can actually show this fact by just considering the bounded operators T_k defined on ℓ_1^2 in [3, Example 4.1] as those on ℓ_2^2 . In fact, for $k \in \mathbb{N}$, consider $Y_k = \mathbb{R}^2$ with the norm

$$\|(x, y)\| = \max \left\{ |x|, |y| + \frac{1}{k}|x| \right\},$$

and $\mathcal{Y} = [\bigoplus_{k=1}^\infty Y_k]_{\ell_\infty}$. Define $T_k \in \mathcal{L}(\ell_2^2, Y_k)$ by

$$T_k(e_1) = \left(-1, 1 - \frac{1}{k}\right) \quad \text{and} \quad T_k(e_2) = \left(1, 1 - \frac{1}{k}\right).$$

Since $B_{\ell_1^2} = D \subset B_{\ell_2^2}$, we have $\|T_k\|_D = 1$ for all $k \in \mathbb{N}$. It follows from the same argument as in [3, Example 4.1] with Theorem 3.2 and Proposition 4.3 that (ℓ_2^2, Y) fails to have the *BPBp* on $D = B_{\ell_1^2}$.

In Section 2, we show that the *BPBp* holds for bounded linear functionals on arbitrary bounded closed convex sets. Using this, we sharpen Stegall's optimization principles [23] in the sense of the *BPBp*. In Section 3, we show that for all finite dimensional Banach spaces X and Y the pair (X, Y) has the *BPBp* on every bounded closed convex subset D of X , and also that for a Banach space Y with property (β) the pair (X, Y) has the *BPBp* on an every bounded closed absolutely convex subset D of an arbitrary Banach space X . For a bounded closed convex subset D of X with positive modulus convexity we get that the pair (X, Y) has the *BPBp* on

D for every Banach space Y . We further prove that for an Asplund space X and for a locally compact Hausdorff L , the pair $(X, C_0(L))$ has the *BPBp* on every bounded closed absolutely convex subset D of X . In Section 4, we study the stability of the *BPBp* for the ℓ_1 -sum or ℓ_∞ -sum of a family of Banach spaces.

2. LINEAR FUNCTIONALS ATTAINING THEIR SUPREMA ON BOUNDED CLOSED CONVEX SETS

We begin by recalling Ekeland's variational principle [13, 15], which can be stated as follows:

Theorem 2.1 (Ekeland). *Let $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous and bounded below function on a Banach space X . Then given $\epsilon > 0$ and $\delta > 0$, there exists $x_1 \in X$ such that $f(x_1) < f(x) + \epsilon\|x - x_1\|$ for every $x \in X$ with $x \neq x_1$. Moreover if $f(x_0) < b + \frac{\delta}{2}$, where $b = \inf\{f(x) : x \in X\}$, then x_1 can be chosen so that $\|x_0 - x_1\| < \frac{\delta}{\epsilon}$.*

The proof of Theorem 7.41 in [13] actually gives the following:

Theorem 2.2. *Let D be a bounded convex closed subset of a Banach space X . Given $\epsilon > 0$ and $\delta > 0$, if $f \in X^*$ and $x_0 \in D$ such that*

$$f(x_0) > \sup\{f(x) : x \in D\} - \frac{\delta}{2},$$

then there exist $g \in X^$ and $x_1 \in D$ satisfying*

$$g(x_1) = \sup\{g(x) : x \in D\}, \quad \|f - g\| \leq \epsilon \quad \text{and} \quad \|x_1 - x_0\| \leq \frac{\delta}{\epsilon}.$$

It is trivial that Theorem 2.2 is not true any more for a complex Banach space ([20]).

Corollary 2.3. *Let $\epsilon > 0$ be given. If $f \in S_{X^*}$ and $x_0 \in S_X$ satisfy that*

$$|1 - f(x_0)| < \frac{\epsilon^2}{4},$$

then there exist $g \in S_{X^}$ and $z \in S_X$ such that*

$$g(z) = 1, \quad \|f - g\| \leq \epsilon \quad \text{and} \quad \|x_0 - z\| \leq \epsilon.$$

Proof. Apply Theorem 2.2 with $\delta = \frac{\epsilon^2}{2}$ and $\epsilon' = \frac{\epsilon}{2}$ and we can choose $\|x^*\| \leq \frac{1}{2}\epsilon$ and $z \in S_X$ so that $f + x^*$ attains its norm at z . Set $g = (f + x^*)/\|f + x^*\|$. Then

$$\begin{aligned} \|f - g\| &\leq \|f - (f + x^*)\| + \|(f + x^*) - g\| \leq \frac{1}{2}\epsilon + \|f + x^*\| \cdot \left|1 - \frac{1}{\|f + x^*\|}\right| \\ &\leq \frac{1}{2}\epsilon + |\|f + x^*\| - 1| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Also we get $|g(z)| = 1$ and $\|x_0 - z\| \leq \frac{\epsilon^2}{2}/\frac{\epsilon}{2} = \epsilon$. □

We can also obtain the following theorem for a bounded linear functional, which is analogous to Stegall's nonlinear form [23] of Bourgain's result mentioned in Introduction.

Theorem 2.4. *Let D be a bounded closed convex set in a Banach space X . Given $0 < \epsilon < 1/4$ and $f \in X^*$, there exist $x^* \in X^*$ and $x_0 \in D$ such that both $f + x^*$ and $f + |x^*|$ attain their suprema simultaneously at x_0 and $\|x^*\| \leq \epsilon$. Moreover $(f + x^*)(x_0) = (f + |x^*|)(x_0)$.*

Proof. We may assume $D \subset B_X$ and $\|f\|_D = 1$. By the Bishop-Phelps theorem, there exists $x^* \in X^*$ such that $f + x^*$ attains its supremum at $x_0 \in D$ and $\|x^*\| \leq \frac{\epsilon}{2}$. If $f(x_0) + x^*(x_0) \geq f(x) + |x^*(x)|$ for every $x \in D$, we are done. Otherwise, there exists $y \in D$ such that $f(y) + |x^*(y)| > f(x_0) + x^*(x_0)$. Clearly $x^*(y) < 0$, and

$$f(y) - x^*(y) > f(x_0) + x^*(x_0).$$

Let $s = \sup_{x \in D} \{f(x) - x^*(x)\}$ and $\alpha = s - (f(x_0) + x^*(x_0)) < (1 + \frac{\epsilon}{2}) - (1 - \frac{\epsilon}{2}) = \epsilon$. Choose $y_0 \in D$ so that $f(y_0) - x^*(y_0) > s - \frac{\alpha^2 \epsilon^2}{2}$. By Theorem 2.2, there exists x_1^* such that $(f - x^*) + x_1^*$ attains its supremum at $z_0 \in D$, $\|x_1^*\| \leq \alpha \epsilon$ and $\|y_0 - z_0\| \leq \alpha \epsilon$. Then,

$$\begin{aligned} f(z_0) - x^*(z_0) + x_1^*(z_0) &\geq f(y_0) - x^*(y_0) - \|x_1^*\| \|z_0 - y_0\| + x_1^*(z_0) \\ &> s - \frac{3\alpha^2 \epsilon^2}{2} - \alpha \epsilon \\ &> f(x_0) + x^*(x_0) + \alpha \epsilon, \end{aligned}$$

where the last inequality follows from the definition of α and $0 < \alpha < \epsilon < 1/4$. Set $x_2^* = -x^* + x_1^*$. Clearly, $\|x_2^*\| \leq \epsilon$. We can see that $f + |x_2^*|$ also attains its supremum $f(z_0) + x_2^*(z_0)$ at z_0 on D . Otherwise, there exists $w \in D$ such that

$$f(w) + |x_2^*(w)| > f(z_0) + x_2^*(z_0),$$

which implies that $x_2^*(w) < 0$. Therefore, we have

$$f(w) - x_2^*(w) > f(z_0) + x_2^*(z_0) > f(x_0) + x^*(x_0) + \alpha \epsilon \geq f(w) + x^*(w) + \|x_1^*\|,$$

which implies that $-x_1^*(w) > \|x_1^*\|$. This contradiction shows that

$$f(z_0) + x_2^*(z_0) = f(z_0) + |x_2^*(z_0)| \geq f(x) + |x_2^*(x)|$$

for every $x \in D$. □

By the same argument as in the proof of Theorem 2.2 we can obtain the following: If $f(x_0) < \inf\{f(x) : x \in D\} + \frac{\delta}{2}$, then there exist $g \in X^*$ and $x_1 \in D$ such that

$$g(x_1) = \inf\{g(x) : x \in D\}, \quad \|f - g\| \leq \epsilon \quad \text{and} \quad \|x_1 - x_0\| \leq \frac{\delta}{\epsilon}.$$

Further, we can show in the following that for a bounded closed convex set D the set $\{f : |f| \text{ attains its supremum on } D\}$ is dense in X^* .

Theorem 2.5. *Let D be a bounded closed convex set in a Banach space X . Given $f \in X^*$ and $\epsilon > 0$, there exists $x^* \in X^*$ such that $|f + x^*|$ attains its supremum on D and $\|x^*\| \leq \epsilon$. Moreover, if D is symmetric, and $f(x_0) > \|f\|_D - \frac{\delta}{2}$ for some $x_0 \in D$ and $\delta > 0$, then x^* and $x_1 \in D$ can be chosen so that $\|x^*\| \leq \epsilon$, $\|x_0 - x_1\| \leq \frac{\delta}{\epsilon}$, and $|f + x^*|$ attains its supremum at x_1 on D .*

Proof. We may assume that D is a bounded closed convex subset of B_X . Let $s = \sup_D f$. We now consider three cases.

1°. Suppose that $s > |\inf_D f|$. Set $\eta = s - |\inf_D f| > 0$. By Theorem 2.4, we can choose $\|x^*\| \leq \min\{\frac{\eta}{2}, \epsilon\}$ so that both $(f + x^*)$ and $(f + |x^*|)$ attain their suprema at $x_0 \in D$ and $(f + x^*)(x_0) = (f + |x^*|)(x_0)$. Since $f(x_0) + x^*(x_0) \geq f(x) + |x^*(x)|$ for every $x \in D$, we have $f(x_0) + x^*(x_0) \geq s$. Therefore, for every $x \in D$ we have

$$-f(x) - x^*(x) \leq -f(x) + |x^*(x)| \leq s - \eta + \frac{\eta}{2} < s \leq f(x_0) + x^*(x_0),$$

which implies that $|f(x) + x^*(x)| \leq f(x_0) + x^*(x_0) = |(f + x^*)(x_0)|$ for every $x \in D$.

2°. Suppose that $s = |\inf_D f|$. By Theorem 2.4, we can choose x^* so that $f + x^*$ attains its supremum at $x_0 \in D$ and $\|x^*\| \leq \frac{\epsilon}{2}$. If $|f + x^*|(x) \leq f(x_0) + x^*(x_0)$ for every $x \in D$, we are done. Otherwise, there exists $y \in D$ such that $|f + x^*|(y) > f(x_0) + x^*(x_0)$. Clearly, $(f + x^*)(y) < 0$ and

$$-f(y) - x^*(y) > f(x_0) + x^*(x_0) \geq f(x) + x^*(x)$$

for every $x \in D$. Set $g = -f - x^*$. Then we have $g(y) > -g(x_0) \geq -g(x)$ for every $x \in D$, which means that $\sup_D g > |\inf_D g|$. It follows from the case 1° that there exist $y^* \in X^*$ and $y_0 \in D$ such that $\|y^*\| \leq \frac{\epsilon}{2}$ and $|g + y^*|(x) \leq (g + y^*)(y_0)$ for every $x \in D$. Therefore,

$$|f + (x^* - y^*)|(x) = |g + y^*|(x) \leq (g + y^*)(y_0) = (-f - x^* + y^*)(y_0) \leq |f + (x^* - y^*)|(y_0).$$

for every $x \in D$ and $\|x^* - y^*\| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \leq \epsilon$.

3° Suppose that $\sup_D f < |\inf_D f|$. We can prove this case by applying the case 1° to $(-f)$.

Further, if D is symmetric, we note that $\|f\|_D = \sup_D f = |\inf_D f|$. From the assumption and Theorem 2.2, we can choose $x^* \in X^*$ and $x_1 \in D$ so that

$$\sup_D (f + x^*) = (f + x^*)(x_1), \quad \|x^*\| \leq \epsilon \quad \text{and} \quad \|x_1 - x_0\| \leq \frac{\delta}{\epsilon}.$$

Since $(f + x^*)(x_1) = \|f + x^*\|_D$, it means that $|f + x^*|$ attains its supremum at x_1 on D . \square

If D is not symmetric in the above theorem, we can hardly choose $x_1 \in D$ satisfying $\|x_0 - x_1\| \leq \frac{\delta}{\epsilon}$ and also that $|f + x^*|$ attains its supremum at x_1 on D . Indeed, for $f \in S_{X^*}$ which does not attain its norm and for $0 < \epsilon < \frac{1}{3}$, we let

$$S_1 = \{x \in B_X : f(x) \geq 1 - \epsilon^2\}, \quad T = B_X \cap \ker f, \quad \text{and} \quad S_2 = \overline{(-S_1 + T)}.$$

Note that S_2 is a closed bounded convex subset of $2B_X$. We set $D = \overline{\text{co}}(S_1 \cup S_2)$. Clearly, $\|f\|_D = 1$ and we can see easily that there exists $x_0 \in D \cap S_X$ such that $f(x_0) > 1 - \frac{\epsilon^2}{2}$.

We claim that for every $\|x^*\| \leq \epsilon$ the function $|f + x^*|$ cannot attain its supremum on D at any point $z \in D$ with $\|z - x_0\| \leq \epsilon$. Otherwise, there exists $\|x^*\| \leq \epsilon$ such that the function $|f + x^*|$ attains its supremum on D at some point $z \in D$ with $\|z - x_0\| \leq \epsilon$. Since

$$(f + x^*)(z) = (f + x^*)(x_0) + (f + x^*)(z - x_0) \geq 1 - \frac{\epsilon^2}{2} - \epsilon - \epsilon(1 + \epsilon) > 0,$$

we can deduce that $(f + x^*)(z) = |(f + x^*)(z)|$ is $\sup_D (f + x^*)$. Choose a sequence $\{z_n\}$ in $\text{co}(S_1 \cup S_2)$ converging to z . For each $n \in \mathbb{N}$ we can write $z_n = (1 - \lambda_n)x_n + \lambda_n y_n$, where $x_n \in S_1$, $y_n \in S_2$ and $0 \leq \lambda_n \leq 1$. An easy computation shows that

$$(f + x^*)(z) \geq 1 - 2\epsilon - \frac{3\epsilon^2}{2} \quad \text{and} \quad (f + x^*)(y_n) \leq -1 + \epsilon^2 + 2\epsilon.$$

It follows from these inequalities that for each $n \in \mathbb{N}$

$$\begin{aligned}
(1 + \epsilon)\|z - z_n\| &> |(f + x^*)(z - z_n)| = |(f + x^*)(z) - (f + x^*)((1 - \lambda_n)x_n + \lambda_n y_n)| \\
&= (1 - \lambda_n) |(f + x^*)(z) - (f + x^*)(x_n)| + \lambda_n |(f + x^*)(z) - (f + x^*)(y_n)| \\
&\geq \lambda_n \left((1 - 2\epsilon - \frac{3\epsilon^2}{2}) - (-1 + \epsilon^2 + 2\epsilon) \right) = \lambda_n (2 - 4\epsilon - \frac{5\epsilon^2}{2}) \geq 0.
\end{aligned}$$

Therefore, we have that for each n

$$0 \leq \lambda_n \leq \frac{(1 + \epsilon)\|z - z_n\|}{(2 - 4\epsilon - \frac{5\epsilon^2}{2})},$$

which implies that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. It means that $z \in S_1$.

Now we recall that $f + x^*$ attains its supremum on D , but f doesn't attain its supremum on D because $\sup_{B_X} f = \sup_D f$ and f doesn't attain its norm. Hence $x^* \neq \alpha f$ for any $\alpha \in \mathbb{R}$, and there exists $w \in T$ with $x^*(w) < 0$ and $\|w\| < \epsilon$. Since $-z + w \in (-S_1 + T) \subset D$, we obtain

$$\sup_D |f + x^*| \geq |(f + x^*)(-z + w)| > (f + x^*)(z) = |(f + x^*)(z)|,$$

which is a contradiction.

3. OPERATORS ATTAINING THEIR SUPREMA IN NORM ON BOUNDED CLOSED CONVEX SETS

It was shown in [2] that for all Banach spaces X and Y of finite dimension (X, Y) has the *BPBp* for B_X . We show that it is still true for arbitrary bounded closed convex subsets which are not necessarily symmetric.

Theorem 3.1. *Let X and Y be finite dimensional Banach spaces. Then the pair (X, Y) has the BPBp on every bounded convex closed subset D of X .*

Proof. Otherwise, there exists a bounded closed convex subset D in X satisfying following condition: For some ϵ_0 , we can find $T_n \in \mathcal{L}(X, Y)$ such that for every $T \in \mathcal{L}(X, Y)$ with $\|T_n - T\| \leq \epsilon_0$, there is $x_n^T \in D$ satisfying $\|T_n(x_n^T)\| > \|T_n\|_D - \frac{1}{n}$ and $\text{dist}(x_n^T, NA_D(T)) \geq \epsilon_0$, where $NA_D(T) = \{z \in D : \|T(z)\| = \|T\|_D\}$. By finite dimensionality, we can choose T_n converging to $T_0 \in \mathcal{L}(X, Y)$ and $x_n^{T_0}$ converging to $x_0 \in D$. Then we can easily show that $\|T_0(x_0)\| = \|T_0\|_D$ and $\|x_n^{T_0} - x_0\| \geq \epsilon_0$, which contradicts $\|x_n^{T_0} - x_0\| \rightarrow 0$. \square

J. Lindenstrauss [19] introduced the notion of property β : A Banach space Y is called to have property β if there is $0 \leq \lambda < 1$ and a family $\{(y_\alpha, f_\alpha) \in B_Y \times B_Y^*\}$ such that (i) $f_\alpha(x_\alpha) = \|x_\alpha\| = 1$ (ii) $|f_\alpha(y_\beta)| \leq \lambda$ for $\alpha \neq \beta$ (iii) $\|y\| = \sup_\alpha |f_\alpha(y)|$.

He showed that if a Banach space Y has property (β) , then the set of all norm attaining operators from X into Y is dense in $\mathcal{L}(X, Y)$ for every Banach space X . In 1982, J. Partington [21] proved rather a surprising result that every Banach space can be renormed to have property (β) . Acosta et al. [2] showed that if Y has property (β) , then (X, Y) has the *BPBp* on B_X for every Banach space X . Now we prove that it is still true for bounded closed convex subsets of X .

Theorem 3.2. *Let Y be a Banach space with property (β) and D be a bounded closed convex set in X . Then (X, Y) has the BPp on D . Moreover, if D is symmetric, then (X, Y) has the BPBp on D .*

Proof. Without loss of generality we may assume $D \subseteq B_X$, and first consider the case where D is symmetric. Let

$$0 < \epsilon < \frac{1-\lambda}{2+\lambda} \quad \text{and} \quad \frac{\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon} \leq \eta \leq \frac{2\epsilon(2+\lambda)}{1-2\epsilon-\lambda-\lambda\epsilon}.$$

Assume that $T \in \mathcal{L}(X, Y)$, $\|T\|_D = 1$, $\|T\| = M$ and $\|Tx_0\| > 1 - \frac{\epsilon^2}{2}$ for some $x_0 \in D$. Then we can choose α_0 so that $|(T^*f_{\alpha_0})(x_0)| = |f_{\alpha_0}(Tx_0)| > 1 - \frac{\epsilon^2}{2}$. By Theorem 2.5, there exist $g \in X^*$ and $z_0 \in D$ such that

$$|g(z_0)| = \|g\|_D, \quad \|g - T^*f_{\alpha_0}\| \leq \epsilon, \quad \text{and} \quad \|x_0 - z_0\| \leq \epsilon.$$

Since $\|g - T^*f_{\alpha_0}\|_D \leq \|g - T^*f_{\alpha_0}\| \leq \epsilon$, we have $1 - 2\epsilon \leq 1 - \frac{\epsilon^2}{2} - \epsilon \leq \|g\|_D \leq 1 + \epsilon$. Define $T_0 \in \mathcal{L}(X, Y)$ by

$$T_0(x) = T(x) + ((1+\eta)g(x) - T^*f_{\alpha_0}(x))y_{\alpha_0}.$$

Clearly

$$\|T_0^*f_{\alpha_0}\|_D = (1+\eta)\|g\|_D \geq (1+\eta)(1-2\epsilon).$$

For $\alpha \neq \alpha_0$,

$$\begin{aligned} \|T_0^*f_{\alpha}\|_D &\leq \|T^*f_{\alpha}\|_D + \lambda(\|g - T^*f_{\alpha_0}\|_D + \eta\|g\|_D) \\ &\leq 1 + \lambda(\epsilon + \eta(1+\epsilon)) \\ &\leq (1+\eta)(1-2\epsilon), \end{aligned}$$

where the last inequality follows from $\eta \geq \frac{\epsilon(\lambda+2)}{1-2\epsilon-\lambda-\lambda\epsilon}$. Since $\|T_0^*f_{\alpha}\|_D \leq \|T_0^*f_{\alpha_0}\|_D$ for all $\alpha \neq \alpha_0$, we have

$$\|T_0\|_D = \sup_{\alpha} \{\|T_0^*f_{\alpha}\|_D\} = \|T_0^*f_{\alpha_0}\|_D = f_{\alpha_0}(T_0z_0) \leq \|T_0z_0\| \leq \|T_0\|_D.$$

Therefore, T_0 attains its supremum at $z_0 \in D$ with $\|x_0 - z_0\| \leq \epsilon$ and

$$\|T - T_0\| \leq \|g - T^*f_{\alpha_0}\| + \eta\|g\| \leq \epsilon + \eta(M + \epsilon) \leq \epsilon \left(1 + \frac{2(2+\lambda)(M+\epsilon)}{1-2\epsilon-\lambda-\lambda\epsilon}\right).$$

For a bounded closed convex subset D , by Theorem 2.5, we can choose $g \in X^*$ and $z_0 \in D$ so that

$$\|g\|_D = |g(z_0)| \quad \text{and} \quad \|g - T^*f_{\alpha_0}\| \leq \epsilon.$$

The rest of proof follows similarly. □

Recall that the modulus of convexity of a Banach space X is defined on B_X by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in B_X, \quad \|x-y\| \geq \epsilon \right\}.$$

We can naturally extend this notion for a bounded closed absorbing convex set D . We define $\delta_D(\epsilon)$ for $0 < \epsilon < 1$ by

$$\delta_D(\epsilon) = \inf \left\{ \frac{1}{2}\rho_D(x) + \frac{1}{2}\rho_D(y) - \rho_D\left(\frac{x+y}{2}\right) : x, y \in D, \quad \rho_D(x-y) \geq \epsilon \right\},$$

where ρ_D is the Minkowski functional of D , that is, $\rho_D(x) = \inf\{\lambda > 0 : x \in \lambda D\}$.

In the following we get such a general result that for a bounded closed absorbing convex subset D of X with positive modulus convexity, the pair (X, Y) has *BPBP* on D for every Banach space Y .

Theorem 3.3. *Let X and Y be (real or complex) Banach spaces and D be a bounded closed absorbing convex subset of B_X such that $\delta_D(\epsilon) > 0$ for every $0 < \epsilon < \frac{1}{2}$. If $T \in S_{\mathcal{L}(X, Y)}$ and $x_1 \in D$ satisfy*

$$\|Tx_1\| > \|T\|_D - \epsilon^3 \delta_D(\epsilon),$$

for sufficiently small ϵ relatively to $\|T\|_D$, then there exist $S \in \mathcal{L}(X, Y)$ and $z \in D$ such that $\|Sz\| = \|S\|_D$, $\|S - T\| < \frac{4\epsilon^2}{1-\epsilon}$ and $\|x_1 - z\| \leq \rho_D(x_1 - z) < \frac{\epsilon}{1-\epsilon}$.

Proof. Let $T_1 = T$. Choose $f_1 \in S_{Y^*}$ so that

$$f_1(T_1 x_1) = \|T_1 x_1\| > \|T_1\|_D - \epsilon^3 \delta_D(\epsilon).$$

Inductively choose $\{T_k\}_{k=2}^\infty$, $\{x_k\}_{k=2}^\infty \subseteq D$, and $\{f_k\}_{k=2}^\infty \subseteq S_{Y^*}$ satisfying

$$T_k(x) = T_{k-1}(x) + \epsilon^k f_{k-1}(T_{k-1}x) T_{k-1}x_{k-1},$$

$$\|T_k x_k\| > \|T_k\|_D - \epsilon^{k+2} \delta_D(\epsilon^k), \quad \rho_D(x_k) = 1,$$

$$f_{k-1}(T_{k-1}x_k) = |f_{k-1}(T_{k-1}x_k)|,$$

$$\text{and } f_k(T_k x_k) = \|T_k x_k\|.$$

Since $\|T_k\| < 2$ and $\|T_{k+1} - T_k\| \leq 2\epsilon^{k+1}\|T_k\|$ for every k , $\{T_k\}_{k=1}^\infty$ converges to S and $\|T - S\| \leq \frac{4\epsilon^2}{1-\epsilon}$. An upper bound of $\|T_{k+1}\|_D$ is

$$\begin{aligned} \|T_{k+1}\|_D &< \|T_{k+1}x_{k+1}\| + \epsilon^{k+3} \delta_D(\epsilon^{k+1}) \\ &\leq \|T_k\|_D + \epsilon^{k+1} |f_k(T_k x_{k+1})| \cdot \|T_k\|_D + \epsilon^{k+3} \delta_D(\epsilon^{k+1}). \end{aligned}$$

A lower bound is

$$\begin{aligned} \|T_{k+1}\|_D &\geq \|T_{k+1}x_k\| = \|T_k x_k\| \cdot |1 + \epsilon^{k+1} f_k(T_k x_k)| \\ &> (\|T_k\|_D - \epsilon^{k+2} \delta_D(\epsilon^k)) (1 + \epsilon^{k+1} (\|T_k\|_D - \epsilon^{k+2} \delta_D(\epsilon^k))) \\ &> \|T_k\|_D + \epsilon^{k+1} \|T_k\|_D^2 - 2\epsilon^{2k+3} \delta_D(\epsilon^k) \|T_k\|_D - \epsilon^{k+2} \delta_D(\epsilon^k). \end{aligned}$$

Combining these two bounds yields

$$|f_k(T_k x_{k+1})| > \|T_k\|_D - 2\epsilon^{k+2} \delta_D(\epsilon^k) - \frac{\epsilon \delta_D(\epsilon^k) + \epsilon^2 \delta_D(\epsilon^{k+1})}{\|T_k\|_D}.$$

Since $\delta_D(\epsilon^k) \geq \delta_D(\epsilon^{k+1})$ and $\|Tx\| \leq \rho_D(x) \|T\|_D$ for every $x \in D$, we have

$$\begin{aligned} \rho_D\left(\frac{x_{k+1} + x_k}{2}\right) \|T_k\|_D &\geq \left\|T_k\left(\frac{x_{k+1} + x_k}{2}\right)\right\| \geq \frac{1}{2} \operatorname{Re}(f_k T_k x_{k+1} + f_k T_k x_k) \\ &\geq \|T_k\|_D - \frac{1}{2} \left(2\epsilon^{k+2} \delta_D(\epsilon^k) + \frac{\epsilon \delta_D(\epsilon^k) + \epsilon^2 \delta_D(\epsilon^{k+1})}{\|T_k\|_D} + \epsilon^{k+2} \delta_D(\epsilon^k)\right) \\ &\geq \|T_k\|_D - \delta_D(\epsilon^k) \left(\frac{3}{2} \epsilon^{k+2} + \frac{\epsilon + \epsilon^2}{2\|T_k\|_D}\right). \end{aligned}$$

It follows from $\|T_k - T\| < \frac{4\epsilon^2}{1-\epsilon} < 4\epsilon$,

$$\|T\|_D - 4\epsilon < \|T_k\|_D < \|T\|_D + 4\epsilon$$

$$\rho_D \left(\frac{x_{k+1} + x_k}{2} \right) > 1 - \delta_D(\epsilon^k) \left(\frac{\frac{3}{2}\epsilon^{k+2} + \frac{\epsilon + \epsilon^2}{2\|T\|_D - 8\epsilon}}{\|T\|_D - 4\epsilon} \right).$$

Since ϵ is sufficiently small relatively to $\|T\|_D$, then we have

$$\begin{aligned} \rho_D \left(\frac{x_{k+1} + x_k}{2} \right) &> 1 - \delta_D(\epsilon^k) \\ &= \frac{1}{2}\rho_D(x_{k+1}) + \frac{1}{2}\rho_D(x_k) - \delta_D(\epsilon^k), \end{aligned}$$

which implies that $\rho_D(x_{k+1} - x_k) \leq \epsilon^k$. Hence $\{x_k\}_{k=1}^\infty$ converges in norm to $z \in D$. We can also see easily that $\|x_1 - z\| \leq \rho_D(x_1 - z) \leq \frac{\epsilon}{1-\epsilon}$, $\|T - S\| \leq \frac{4\epsilon^2}{1-\epsilon}$ and $\|Sz\| = \|S\|_D$. □

Corollary 3.4 ([18]). *Let X be a uniformly convex Banach space and Y be a Banach space. Then (X, Y) has the BPBp on B_X .*

Remark 3.5. It is easy to notice that the BPBp is an isometric property, but not isomorphic. On the other hand, the BPBp still holds on the image of an isomorphism in the following sense. Let Ψ be an isomorphism from a Banach space X into a Banach space Z . We can see that if (X, Y) has the BPBp on B_X , then $(\Psi(X), Y)$ has the BPBp on $D = \Psi(B_X)$. Further, if Y is injective, then (Z, Y) has the BPBp on D . We recall that a Banach space Z is called injective if for every Banach space X and for every subspace W of X , every operator from W into Z can be extended to an operator from X into Z preserving its norm.

A Banach space X is called an Asplund space if the set of all points of U where f is Fréchet differentiable is dense G_δ -subset of U for every real-valued convex continuous function f defined on an open convex subset $U \subseteq X$. Equivalently every w^* -compact subset of (X^*, w^*) is $\|\cdot\|$ -fragmentable. Here we say a subset C of (X^*, w^*) is $\|\cdot\|$ -fragmentable if for every nonempty bounded subset $A \subset C$ and for every $\epsilon > 0$, there is a nonempty w^* -open neighborhood $V \subset X^*$ such that $A \cap V$ is nonempty and has $\|\cdot\|$ -diameter less than ϵ ([13]). Recall that an operator $T \in \mathcal{L}(X, Y)$ is called by an Asplund operator if it factors through an Asplund space. That is, there are an Asplund space Z and operators $T_1 \in \mathcal{L}(X, Z)$ and $T_2 \in \mathcal{L}(Z, Y)$ such that $T = T_2 \circ T_1$. For example, every weakly compact operator is an Asplund operator since it factors through a reflexive space, so that a rank one operator is an Asplund operator. We also note that the family of Asplund operators is an operator ideal, hence the sum of two Asplund operators or the composition of an operator with an Asplund operator is again an Asplund operator.

It was shown in [5] that the BPBp on B_X holds for an Asplund operator from X into $C_0(L)$. We can extend this result to a symmetric bounded closed convex subset $D \subset B_X$. Some modifications of [5, Lemma 2.3 and Theorem 2.4] are just needed, but we give the details for the sake of completeness.

Lemma 3.6. *Let D be a symmetric bounded closed convex subset of B_X and $T \in \mathcal{L}(X, Y)$ be an Asplund operator with $\|T\|_D = 1$ and $\|T\| \geq M \geq 1$. If*

$$\|T(x_0)\| > 1 - \frac{\epsilon^2}{4M},$$

for some $x_0 \in D$, then for every norming set $B \subseteq B_{Y^}$ and $0 < \epsilon \leq \frac{M}{2}$, there exist $x^* \in X^*$, $u_0 \in D$ and a w^* -open neighborhood U in X^* such that*

$$|x^*(u_0)| = 1 = \|x^*\|_D, \quad \|x_0 - u_0\| < \epsilon \quad \text{and} \quad \|z^* - x^*\| < 4\epsilon,$$

for every $z^ \in U \cap T^*(B)$.*

Proof. Since B is a norming set, we can choose $y_0^* \in B$ such that

$$|y^*(Tx_0)| = |T^*y_0^*(x_0)| > 1 - \frac{\epsilon^2}{4M}.$$

Define a w^* -open neighborhood in X^* by

$$U_1 = \left\{ z^* \in X^* : |z^*(x_0)| > 1 - \frac{\epsilon^2}{4M} \right\}.$$

Since $T^*y_0^* \in T^*(B) \cap U_1$, we get $T^*(B) \cap U_1 \neq \emptyset$. Since $T^*(B)$ is $\|\cdot\|$ -fragmentable, ([5]), we can find a w^* -open neighborhood $U_2 \subset X^*$ such that

$$U \cap T^*(B) \neq \emptyset \quad \text{and} \quad \text{diam}(U \cap T^*(B)) < \epsilon,$$

where $U = U_1 \cap U_2$. Now fix $z_0^* \in U \cap T^*(B)$. Write $z_0^* = T^*(w_0^*)$ for some $w_0^* \in B \subset B_{Y^*}$. We can see $\|z_0^*\| \leq M$ and

$$\|z_0^*\|_D = \sup_{x \in D} |T^*w_0^*(x)| = \sup_{x \in D} |w_0^*(Tx)| \leq \|T\|_D \leq 1,$$

which implies that $|z_0^*(x_0)| > 1 - \frac{\epsilon^2}{4M} \geq \|z_0^*\|_D - \frac{\epsilon^2}{4M}$. By Theorem 2.5, we can choose $x^* \in X^*$ and $u_0 \in D$ such that

$$\|x^* - z_0^*\| < \frac{\epsilon}{2M}, \quad \|x_0 - u_0\| < \epsilon \quad \text{and} \quad |x^*(u_0)| = \|x^*\|_D.$$

It also follows from an easy computation that

$$1 - \frac{\epsilon}{M} \leq \|x^*\|_D \leq 1 + \frac{\epsilon}{2M}.$$

Let $k = \frac{1}{\|x^*\|_D}$. Clearly,

$$\frac{2M}{2M + \epsilon} \leq k \leq \frac{M}{M - \epsilon}$$

and

$$\begin{aligned} \|kx^* - z_0^*\| &\leq \|kx^* - x^*\| + \|x^* - z_0^*\| \leq (k - 1)\|x^*\| + \frac{\epsilon}{2M} \\ &\leq \frac{\epsilon}{M - \epsilon} \left(M + \frac{\epsilon}{2M} \right) + \frac{\epsilon}{2M} \leq 2\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq 3\epsilon, \end{aligned}$$

Since $|kx^*(u_0)| = 1$ from the choice of k , kx^* satisfies the desired properties. Moreover,

$$\|kx^* - z^*\| \leq 3\epsilon + \epsilon = 4\epsilon,$$

for every $z^* \in U \cap T^*(B)$. □

Theorem 3.7. *For a symmetric bounded closed convex subset $D \subseteq B_X$ and a locally compact Hausdorff space L , let $T : X \rightarrow C_0(L)$ be an Asplund operator with $\|T\|_D = 1$ and $\|T\|_D = M \geq 1$. Given $0 < \epsilon < \frac{M}{2}$, if $x_0 \in D$ satisfies that*

$$\|Tx_0\| > 1 - \frac{\epsilon^2}{4M},$$

then there exist an Asplund operator $S : X \rightarrow C_0(L)$ and a point $u_0 \in D$ such that

$$\|S\|_D = \|Su_0\| = 1, \quad \|x_0 - u_0\| < \epsilon \quad \text{and} \quad \|T - S\| < 4\epsilon.$$

Proof. Define $\delta : L \rightarrow C_0(L)^*$ by $\delta(s)(f) = f(s)$ for $f \in C_0(L)$ and $s \in L$. It is easy to check that $\phi = T^* \circ \delta : L \rightarrow X^*$ is w^* -continuous. Since $\{\delta(s) : s \in L\}$ is a norming set in $B_{C_0^*(L)}$, we can find a w^* -open neighborhood U and $x^* \in X^*$ by Lemma 3.6. Here we have $T(x)(s) = \phi(s)(x)$ and $T^*(B) = \phi(L)$. Since $U \cap T^*(B) \neq \emptyset$, there is $s_0 \in L$ such that $\phi(s_0) \in U$. Consider the set

$$W = \{s \in L : \phi(s) \in U\},$$

which is an open neighborhood of s_0 due to the w^* -continuity of ϕ . By Urysohn's lemma there exists a continuous function $f : L \rightarrow [0, 1]$ satisfying

$$f(s_0) = 1 \quad \text{and} \quad \text{supp}(f) \subset W.$$

Define a linear operator $S : X \rightarrow C_0(L)$ by

$$S(x)(s) = f(s)x^*(x) + (1 + f(s))T(x)(s).$$

Define $S_2 \in \mathcal{L}(C_0(L), C_0(L))$ by $S_2(h) = (1 + f)h$. Then $S(x) = f \cdot x^*(x) + S_2(T(x))$, hence S is an Asplund operator. It follows easily that $\|S\|_D \leq 1$ and $|S(u_0)(s_0)| = |y^*(u_0)| = 1$, which shows that S attains its supremum at u_0 on D and $\|u_0 - x_0\| < \epsilon$. For an upper bound of $\|T - S\|$,

$$\begin{aligned} \|T - S\| &= \sup_{x \in B_X} \|Tx - Sx\| = \sup_{x \in B_X} \sup_{s \in L} |f(s)| \cdot |T(x)(s) - x^*(x)| \\ &= \sup_{s \in W} \sup_{x \in D} |\phi(s)(x) - x^*(x)| \leq \sup_{s \in W} \|\phi(s) - x^*\| \leq 4\epsilon, \end{aligned}$$

where the last inequality is derived from $\phi(s) \in U \cap T^*(B)$ and Lemma 3.6. This completes the proof. \square

Corollary 3.8. *For any (real) Asplund space X and any locally compact Hausdorff space L , the pair $(X, C_0(L))$ has the BPBp on a symmetric bounded closed convex set D of X .*

Remark 3.9. By the remark after Theorem 2.5, without the symmetry of D we can show only that there exists an Asplund operator $X \rightarrow C_0(L)$ such that

$$u_0 \in D, \quad \|S\|_D = 1 = \|S(u_0)\| \quad \text{and} \quad \|T - S\| < 4\epsilon,$$

by modifying the proof of Lemma 3.6 and by the first part of Theorem 2.5.

4. STABILITY OF THE BISHOP-PHELPS-BOLLOBÁS PROPERTY ON DIRECT SUMS

In order to compare the function $\eta(\epsilon)$ appearing in the definition of the BPBp for different pairs (X, Y) , the notion of $\eta(X, Y)(\epsilon)$ was introduced in [3]. We now generalize it to a bounded closed convex subset D of B_X .

Definition 4.1. Let X and Y be (real or complex) Banach spaces. For a bounded closed convex subset $D \subseteq B_X$ and $T \in \mathcal{L}(X, Y)$,

$$\Pi_D(X, Y) = \{(x, T) : x \in D, \|T(x)\| = \|T\|_D = 1\}$$

$$\eta_D(X, Y)(\epsilon) = \inf\{1 - \|Tx\| : x \in D, \|T\|_D = 1, \text{dist}((x, T), \Pi_D(X, Y)) \geq \epsilon\},$$

where $\text{dist}((x, T), \Pi_D(X, Y)) = \inf\{\max\{\|x - y\|, \|T - S\|\} : (y, S) \in \Pi_D(X, Y)\}$.

It is clear the pair (X, Y) has the *BPBP* on D if and only if $\eta_D(X, Y)(\epsilon) > 0$ for every $0 < \epsilon < 1$. If a function $\epsilon \mapsto \eta_D(\epsilon)$ is valid in the definition of the *BPBP* on D for the pair (X, Y) , then $\eta_D(\epsilon) \leq \eta_D(X, Y)(\epsilon)$. In other words, $\eta_D(X, Y)(\epsilon)$ is the largest function we can find to ensure that (X, Y) has the *BPBP* on D . It is clear that $\eta_D(X, Y)(\epsilon)$ is increasing with respect to ϵ .

Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be families of Banach spaces, $X = (\bigoplus_{i \in I} X_i)_{\ell_1}$ and $Y = (\bigoplus_{j \in J} Y_j)_{\ell_\infty}$. Let E_i and F_j be the natural isometric embeddings of X_i and Y_j into X and Y , respectively and let P_i and Q_j be the canonical projections of norm one from X and Y onto X_i and Y_j , respectively. For $D \subset X$ we let $D_i = \overline{P_i(D)}^{X_i}$ for each $i \in I$.

Payá and Saleh [22] studied the denseness of norm attaining operators from the ℓ_1 -sum of domain space into the ℓ_∞ -sum of range spaces. Their methods in [22] were applied in studying the Bishop-Phelps-Bollobás property for operators on those spaces ([3, 12]). With some suitable condition on D , we have the following analogous results to [3, Theorem 2.1].

Proposition 4.2. *Let $X = (\bigoplus_{i \in I} X_i)_{\ell_1}$, $Y = (\bigoplus_{j \in J} Y_j)_{\ell_\infty}$ and D be a bounded closed convex subset of B_X . Suppose that $D = \overline{\text{co}}(\cup E_i D_i) \subset B_X$. If the pair (X, Y) has the *BPBP* with $\eta_D(\epsilon)$ on D , then the pair (X_i, Y_j) has the *BPBP* on D_i for every $i \in I$ and for every $j \in J$. More precisely,*

$$\eta_D(X, Y)(\epsilon) \leq \eta_{D_i}(X_i, Y_j)(\epsilon), \quad (i \in I, j \in J).$$

Proof. Fix $h \in I$ and $k \in J$. Suppose that $\|T(x_h)\| > 1 - \eta_D(\epsilon)$ for $T \in \mathcal{L}(X_h, Y_k)$, $\|T\|_{D_h} = 1$ and $x_h \in D_h$. Define an operator $\tilde{T} = F_k T P_h \in \mathcal{L}(X, Y)$. It is easy to check that

$$\|\tilde{T}\|_D = \|T\|_{D_h},$$

so that $\|\tilde{T}(E_h x_h)\| > 1 - \eta_D(\epsilon)$. The assumption gives us $(u, \tilde{S}) \in \Pi_D(X, Y)$ such that

$$\|\tilde{S} - \tilde{T}\| < \epsilon \quad \text{and} \quad \|u - E_h x_h\| < \epsilon.$$

Define $S = Q_k \tilde{S} E_h \in \mathcal{L}(X_h, Y_k)$. Then $\|S - T\| \leq \|\tilde{S} - \tilde{T}\| < \epsilon$. For $j \neq k$, we have $Q_j \tilde{T} = 0$ by the definition of \tilde{T} , which implies that

$$\|Q_j \tilde{S}\|_D = \|Q_j \tilde{S} - Q_j \tilde{T}\|_D \leq \|\tilde{S} - \tilde{T}\| < \epsilon.$$

Since the range of \tilde{S} is the ℓ_∞ -sum of Y_j 's, we have $\|Q_k \tilde{S}\|_D = \|\tilde{S}\|_D = 1 = \|\tilde{S}(u)\| = \|Q_k \tilde{S} u\|$. It follows from the assumption $D = \overline{\text{co}}(\cup E_i D_i)$ that every $u \in D$ can be written by

$$u = \sum_{i=1}^{\infty} \lambda_i E_i u_i,$$

where $u_i \in D_i$ and $\sum_{i=1}^{\infty} \lambda_i \leq 1$. Indeed, choose $v^n \in co(\cup E_i D_i)$ converging to u . We can write $v^n = \sum_{i=1}^{\infty} \lambda_i^n E_i v_i^n$ ($v_i^n \in D_i$), where $v_i^n = 0$ and $\lambda_i^n = 0$ except finitely many i 's. Since $E_i D_i \cap E_j D_j = \{0\}$ for $i \neq j$, we have that $\lambda_i^n v_i^n \rightarrow P_i u$ for every i as $n \rightarrow \infty$. By the diagonal argument, up to a subsequence, there exists a sequence $\{\lambda_i\}$ such that $\lambda_i^n \rightarrow \lambda_i \geq 0$ for every i as $n \rightarrow \infty$. By the Fatou lemma we obtain that $\sum_{i=1}^{\infty} \lambda_i \leq 1$. We can also see that there exists $u_i \in D_i$ for every i such that $v_i^n \rightarrow u_i$ as $n \rightarrow \infty$ and $\lambda_i u_i = P_i u$, hence $u = \sum_{i=1}^{\infty} \lambda_i E_i u_i$.

Since $\|Q_k \tilde{S}(u)\| = 1$ and $E_i u_i \in D$ for every i , we have that $\|Q_k \tilde{S}(E_i u_i)\| = 1$ for every i where $\lambda_i \neq 0$ and also that $\sum_{i=1}^{\infty} \lambda_i = 1$. It follows from $\tilde{T} E_i = 0$ for $i \neq h$ that $\|Q_k \tilde{S} E_i\| \leq \|\tilde{T} - \tilde{S}\| \leq \epsilon$ for $i \neq h$. Therefore,

$$1 = \|Q_k \tilde{S} u\| \leq \lambda_h \|Q_k \tilde{S} E_h u_h\| + \epsilon \sum_{i \neq h} \lambda_i \leq \sum \lambda_i = 1,$$

which implies that $\lambda_i = 0$ for $i \neq h$, $\lambda_h = 1$ and $\|S(u_h)\| = \|Q_k \tilde{S} E_h u_h\| = 1 = \|S\|_{D_h}$. Further,

$$\|u_h - x_h\| = \|P_h(u - E_h x_h)\| \leq \|u - E_h x_h\| \leq \epsilon.$$

□

If we fix the domain space X , then the reverse inequality also holds for the ℓ_{∞} -sum of range spaces.

Proposition 4.3. $\eta_D(X, Y) = \inf_{j \in J} \eta_D(X, Y_j)$

Proof. It is enough to prove that $\eta_D(X, Y) \geq \inf_{j \in J} \eta_D(X, Y_j)$. Fix $\epsilon \in (0, 1)$. Let $0 \leq \alpha = \inf_{j \in J} \eta_D(X, Y_j)(\epsilon) < 1$. For $0 < \alpha < 1$, suppose that $\|T x_0\| > 1 - \alpha$ for $x_0 \in D$ and $T \in \mathcal{L}(X, Y)$ with $\|T\|_D = 1$. We can choose $k \in J$ so that $\|Q_k T x_0\| > 1 - \alpha$. Then there exist $S_k : X \rightarrow Y_k$ and $u \in D$ such that

$$\|S_k u\| = \|S_k\|_D = 1, \quad \|S_k - Q_k T\| < \epsilon \quad \text{and} \quad \|x_0 - u\| < \epsilon.$$

Define $S : X \rightarrow Y$ by

$$S = \sum_{j \neq k} F_j Q_j T + F_k S_k.$$

It is easy to check that $(u, S) \in \Pi_D(X, Y)$. Moreover

$$\|T - S\| = \sup_{j \in J} \|Q_j(T - S)\| = \|Q_k T - S_k\| < \epsilon,$$

which means that $\eta_D(X, Y)(\epsilon) \geq \alpha$. □

We now consider the case where X is the ℓ_{∞} -sum of domain spaces X_i .

Proposition 4.4. Let $X = [\bigoplus_{i \in I} X_i]_{\ell_{\infty}}$. Assume that D is a bounded closed convex subset of B_X , $D = \Pi_{i \in I} D_i$ and that there exists $\epsilon_0 > 0$ such that $\frac{\lambda x_i}{\|x_i\|} \in D_i$ for every $x_i \in D_i$ and for every $0 \leq \lambda \leq \epsilon_0$. If the pair (X, Y) has the BPBp on D with $\eta(\epsilon)$, then the pair (X_i, Y) has the BPBp on D_i with $\eta(\epsilon)$ for every $i \in I$. More precisely,

$$\eta_{D_i}(X_i, Y)(\epsilon) \geq \eta_D(X, Y)(\epsilon) \quad \text{for every } i \in I.$$

Proof. Fix $h \in I$. Suppose that $\|T(x_h)\| > 1 - \eta(\epsilon)$ for some $T \in \mathcal{L}(X_h, Y)$ with $\|T\|_{D_h} = 1$ and $x_h \in P_h(D)$. Define $\tilde{T} \in \mathcal{L}(X, Y)$ by $\tilde{T}(u_h, z) = T(u_h)$, where $(u_h, z) \in P_h X \oplus (I - P_h)X$. Then $\|\tilde{T}\|_D = 1$ and $\|\tilde{T}(E_h x_h)\| > 1 - \eta(\epsilon)$. Since $E_h x_h \in D$ and the pair (X, Y) has the *BPBP* on D , for $0 < \epsilon < \epsilon_0$ there exist $\tilde{S} \in \mathcal{L}(X, Y)$ with $\|\tilde{S}\|_D = 1$ and $u \in D$ such that

$$\|\tilde{S}(u)\| = 1, \quad \|\tilde{S} - \tilde{T}\| < \epsilon, \quad \text{and} \quad \|E_h x_h - u\| < \epsilon.$$

Now we define an operator $S \in \mathcal{L}(X_h, Y)$ for $u_h \in X_h$

$$S(u_h) = \tilde{S}(E_h u_h).$$

From $\|E_h x_h - u\| < \epsilon$, we get $\|P_i u\| < \epsilon$ for $i \neq h$. The assumption yields that $\frac{\epsilon_0}{\epsilon} P_i(u) \in D_i$ for $i \neq h$. Let w be the element in D such that $P_h(w) = P_h(u)$ and $P_i(w) = \frac{\epsilon_0}{\epsilon} P_i u$ for $i \neq h$. Then

$$\tilde{S}(w) = \tilde{S}(E_h P_h u) + \sum_{i \neq h} \frac{\epsilon_0}{\epsilon} \tilde{S}(E_i P_i u),$$

hence,

$$\tilde{S}(u) = \left(1 - \frac{\epsilon}{\epsilon_0}\right) \tilde{S}(E_h P_h u) + \frac{\epsilon}{\epsilon_0} \tilde{S}(w).$$

It follows from $\|\tilde{S}\|_D = 1 = \|\tilde{S}(u)\|$ that $\|\tilde{S}(E_h P_h u)\| = \|\tilde{S}(w)\| = 1$. Since $\|S\|_{D_h} \leq \|\tilde{S}\|_D = 1$, S attains its maximum at $P_h u$ on D_h . Moreover,

$$\|S - T\| < \epsilon \quad \text{and} \quad \|x_h - P_h u\| = \|E_h x_h - E_h P_h u\| \leq \|E_h x_h - u\| < \epsilon,$$

so that the proof is completed. \square

Examples satisfying the above assumption on D include $\bigoplus_{i \in I} \lambda_i B_{X_i}$ with $\inf_{i \in I} \lambda_i > 0$ as well as B_X . The case of the ℓ_1 -sum of range spaces follows immediately from [3, Proposition 2.7], so we omit the proof.

Proposition 4.5. *Let D be a bounded closed convex subset of B_X and $Y = [\bigoplus_{j \in J} Y_j]_{\ell_1}$. If the pair (X, Y) has the *BPBP* on D with $\eta(\epsilon)$, then the pair (X, Y_j) also has the *BPBP* on D with $\eta(\epsilon)$ for every $j \in J$. More precisely, for every $j \in J$,*

$$\eta_D(X, Y) \leq \eta_D(X, Y_j)$$

A Banach space X is called a universal *BPB* domain space if for every Banach space Z , the pair (X, Z) has the *BPBP* on B_X . It was proved in [3] that the base field \mathbb{R} or \mathbb{C} is the unique Banach space which is a universal *BPB* domain space in any equivalent renorming. Its proof follows immediately from [3, Lemma 3.2]: Let X be a Banach space containing a non-trivial L -summand and Y be a strictly convex Banach space. If the pair (X, Y) has the *BPBP* on B_X , then Y is uniformly convex. We can extend this result to a bounded closed convex subset D of X . With proposition 4.6 and remark 3.5, using similar argument in [3], we can say that the base field \mathbb{R} or \mathbb{C} is the unique *BPB* domain on every bounded closed convex subset D .

Proposition 4.6. *Let X be a (real) Banach space containing a nontrivial L -summand, i.e. $X = X_1 \oplus X_2$ for some non trivial subspaces X_1 and X_2 . Let D be a bounded closed convex subset of B_X such that $D = \overline{\text{co}}(E_1 D_1 \cup E_2 D_2)$. If Y is a strictly convex space and if the pair (X, Y) has the *BPBP* on D , then Y is a uniformly convex space.*

Proof. To prove that Y is uniformly convex, for every $0 < \epsilon < 1/2$ we need to find $\delta(\epsilon) > 0$ such that $\|y_1\| = 1 = \|y_2\|$ and $\|\frac{y_1+y_2}{2}\| > 1 - \delta(\epsilon)$ implies that $\|y_1 - y_2\| < \epsilon$. By the Bishop-Phelps theorem, we can choose $e_1^* \in X_1^*$ such that

$$\|e_1^*\|_{D_1} = e_1^*(e_1) = 1,$$

for some $e_1 \in D_1$. We can extend (still call it by) $e_1^* \in X^*$ by $e_1^*(x_2) = 0$ for every $x_2 \in X_2$. Similarly we can choose $e_2^* \in X^*$ and $e_2 \in D_2$ as a counterpart of e_1^* and e_1 , respectively. Let $M \geq 4$ be a constant with $\|e_1^*\| < M$ and $\|e_2^*\| < M$. Suppose that $y_1, y_2 \in S_Y$ and $\|\frac{y_1+y_2}{2}\| > 1 - \eta_D(\frac{\epsilon}{2M})$. Define an operator $T \in \mathcal{L}(X, Y)$ by

$$T(u_1, u_2) = e_1^*(u_1)y_1 + e_2^*(u_2)y_2.$$

Then $\|T\| < 2M$ and $\|T(\frac{1}{2}E_1(e_1) + \frac{1}{2}E_2(e_2))\| = \|\frac{y_1+y_2}{2}\| > 1 - \eta_D(\frac{\epsilon}{2M})$. Since $\frac{1}{2}E_1(e_1) + \frac{1}{2}E_2(e_2) \in D$ and since (X, Y) has the *BPBp* on D , there exist $S \in \mathcal{L}(X, Y)$ and $(x_1, x_2) \in D$ such that

$$\|S - T\| < \frac{\epsilon}{2M}, \quad \left\| \frac{1}{2}e_1 - x_1 \right\| + \left\| \frac{1}{2}e_2 - x_2 \right\| < \frac{\epsilon}{2M} \quad \text{and} \quad \|S(x_1, x_2)\| = 1.$$

We can see that $x_1 \neq 0 \neq x_2$. Indeed, if $x_1 = 0$, then

$$1 = \|S(0, x_2)\| \leq \|T(0, x_2)\| + \frac{\epsilon}{2M} \leq |e_2^*(x_2)| + \frac{\epsilon}{M} \leq \frac{1}{2} + \frac{2\epsilon}{M} < 1,$$

which is a contradiction.

Choose $\{z_n\} \subset co(E_1D_1 \cup E_2D_2)$ converging to (x_1, x_2) . Since E_1D_1 and E_2D_2 are convex sets, z_n can be written as $\lambda_n u_n + (1 - \lambda_n)v_n$, where $u_n \in E_1D_1$ and $v_n \in E_2D_2$. Passing to a subsequence we may assume $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Then it is easy to check that $\lambda u_n \rightarrow E_1x_1$ and $(1 - \lambda)v_n \rightarrow E_2x_2$ as $n \rightarrow \infty$, which implies $0 < \lambda < 1$ because $x_1 \neq 0 \neq x_2$. Since E_1D_1 and E_2D_2 are closed, we can see that $\frac{E_1x_1}{\lambda} \in E_1D_1$ and $\frac{E_2x_2}{1-\lambda} \in E_2D_2$.

We claim that $|\lambda - \frac{1}{2}| \leq \frac{1}{2}\epsilon$. Suppose $\lambda < \frac{1}{2} - \frac{1}{2}\epsilon$. Let $\delta = \frac{1}{2} - \lambda > \frac{\epsilon}{2}$ and $x_0 = \frac{1}{2}e_1 - x_1$. Then

$$\frac{x_1}{\lambda} = \frac{e_1}{1-2\delta} + \frac{2x_0}{1-2\delta},$$

Since $|e_1^*(x_0)| \leq \|e_1^*\| \cdot \|x_0\| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}$, we have

$$e_1^*\left(\frac{x_1}{\lambda}\right) = \frac{1}{1-2\delta}e_1^*(e_1) + \frac{2}{1-2\delta}e_1^*(x_0) > \frac{1}{1-2\delta} - \frac{2}{1-2\delta} \cdot \frac{\epsilon}{2} = \frac{1-\epsilon}{1-2\delta} > 1.$$

Since $\frac{x_1}{\lambda} \in D_1$, this contradicts to $\|e_1^*\|_{D_1} = 1$. We can use the same argument for the case $\lambda > \frac{1}{2} + \epsilon$, which completes the claim. Then

$$\left\| e_1 - \frac{x_1}{\lambda} \right\| \leq 2 \left\| \frac{1}{2}e_1 - x_1 \right\| + \left\| 2x_1 - \frac{x_1}{\lambda} \right\| \leq \frac{\epsilon}{M} + \|x_1\| \cdot \left| \frac{2\lambda - 1}{\lambda} \right| \leq \frac{\epsilon}{M} + 4\epsilon.$$

Since

$$1 = \|S(x_1, x_2)\| = \left\| \lambda S\left(\frac{x_1}{\lambda}, 0\right) + (1-\lambda)S\left(0, \frac{x_2}{1-\lambda}\right) \right\|,$$

and since $\|S\|_D = 1$, the inequalities $\|S(\frac{x_1}{\lambda}, 0)\| \leq 1$ and $\|S(0, \frac{x_2}{1-\lambda})\| \leq 1$ combined with the strict convexity of Y yields that $S(\frac{x_1}{\lambda}, 0) = S(0, \frac{x_2}{1-\lambda})$. Moreover

$$\begin{aligned} \|y_1 - y_2\| &= \|T(e_1, 0) - T(0, e_2)\| \\ &\leq \|T(e_1, 0) - S(e_1, 0)\| + \|S(e_1, 0) - S(\frac{x_1}{\lambda}, 0)\| \\ &\quad + \|S(0, \frac{x_2}{1-\lambda}) - S(0, e_2)\| + \|S(0, e_2) - T(0, e_2)\| \\ &\leq 2\|T - S\| + \|S\| \left(\|e_1 - \frac{x_1}{\lambda}\| + \|e_2 - \frac{x_2}{1-\lambda}\| \right) \\ &\leq \epsilon \left(\frac{1}{M} + (2M + \frac{\epsilon}{2M}) \cdot 2(\frac{1}{M} + 4) \right), \end{aligned}$$

where the last inequality follows from $\|S - T\| < \frac{\epsilon}{2M}$ and $\|T\| < 2M$. Therefore Y is uniformly convex. \square

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